Algebraic Manipulations on the AMC 10/12

Guide 1 — Symmetry, Substitution, and Hidden Structure

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1 Introduction

Algebraic manipulation is not so much about doing more algebra as it is about finding a reason to do less. Consider a problem that gives $x + y$ and xy, then asks for $x^3 + y^3$. A familiar approach is to solve a quadratic for x and y, raise both roots to the third power, and simplify. That method works, but it introduces square roots only to have them disappear. The question never needed the individual values of x and y in the first place.

Competition problems on the AMC 10/12 and AIME frequently reward a different way of thinking. Rather than treating every unknown as something that must eventually be solved, we identify the *information* required by the final expression. A symmetric expression usually calls for sums and products. A fraction may become linear after taking reciprocals. A large system may have the same total sum hidden in each equation. An expression full of high powers may be governed by a short recurrence.

These moves are not unrelated tricks. They are instances of one principle: *change the representation until the desired quantity is easy to see.* Each section below develops a small collection of identities, explains why they are useful, and shows how a careful solver might discover them during a contest. The practice section is organized by difficulty and ends with challenges appropriate for students preparing to qualify for, or compete in, the AIME. Try the problems before reading the solutions; the point is to train the instinct for which manipulation to attempt.

2 Foundational Identities and the Target-First Mindset

The most familiar algebraic identities are also among the most powerful, provided we apply them in the right direction. Students are often asked to expand $(a+b)^2$ or factor $a^3 - b^3$, but contest problems may present the expanded pieces and require us to recognize the compact expression that produced them.

Fundamental Identity. For arbitrary real or complex a, b,

$$\begin{aligned} (a+b)^2 &= a^2 + 2ab + b^2, & (a-b)^2 &= a^2 - 2ab + b^2, \\ a^2 - b^2 &= (a-b)(a+b), & a^3 + b^3 &= (a+b)(a^2 - ab + b^2), \\ a^3 - b^3 &= (a-b)(a^2 + ab + b^2), & (a+b)^3 &= a^3 + b^3 + 3ab(a+b). \end{aligned}$$

Notice that the last identity is most useful when read *backward*: if $a + b$ and ab are known, then $a^3 + b^3$ is known even when a and b themselves are not. Likewise, the difference-of-squares formula is an instruction to multiply two known expressions rather than solve for two unknowns.

Example 2.1 (The expression appears after one multiplication)**.** Suppose $x + y = 7$ and $x - y = 3$. Find $x^2 - y^2$.

Solution. We could first obtain $x = 5$ and $y = 2$, but that is more information than necessary. The target already has the factorization

$$x^2 - y^2 = (x-y)(x+y) = 3 \cdot 7 = \boxed{21}.$$

The main step was recognizing the final expression as a product of the two givens. □

Example 2.2 (A shifted product)**.** If $a + b = 9$ and $ab = 14$, evaluate $(a-2)(b-2)$.

Solution. The new factors may look unrelated to the given sum and product, but expanding them creates exactly those quantities:

$$(a-2)(b-2) = ab - 2(a+b) + 4 = 14 - 18 + 4 = \boxed{0}.$$

In fact, the identity tells us that at least one of a, b equals 2. We do not need to decide which one. □

A useful habit is to examine the shape of the answer before manipulating the givens. If the answer is a^3+b^3, think about $(a+b)^3$. If it is a^2-b^2, look for $a+b$ and $a-b$. If it is $(a-1)(b-1)$, expand that product instead of operating on the whole problem. This strategy of moving backward from the target avoids unnecessary steps and saves you time.

3 Symmetric Expressions in Two Variables

An expression is symmetric in x and y if swapping the two variables leaves its value unchanged. Examples include x^2+y^2, xy, and $x^4-5x^3y-5xy^3+y^4$. Symmetric variables don't care which variable is which. Our algebra should reflect that.

Set

$$s=x+y, \qquad p=xy.$$

These two quantities are the natural coordinates for symmetric expressions in two variables. By expanding powers of $x+y$ and collecting terms, we obtain a compact dictionary:

$$\begin{aligned} x^2+y^2 &= s^2-2p,\\ (x-y)^2 &= s^2-4p,\\ x^3+y^3 &= s^3-3ps,\\ x^4+y^4 &= s^4-4ps^2+2p^2,\\ (x+k)(y+k) &= p+ks+k^2. \end{aligned}$$

All of these follow from identities in the previous section. For instance, x^4+y^4 is simply $(x^2+y^2)^2-2x^2y^2$, so we can compute it by first replacing x^2+y^2 with s^2-2p. This is more reliable than memorizing a separate long formula for every degree.

Example 3.1 (Finding a cube without solving a quadratic)**.** Real numbers a,b satisfy $a+b=7$ and $ab=5$. Determine a^3+b^3. *(Theme adapted from Randolph Math League, 2013–14.)*

Solution. We need only the sum and product, which are already given. Therefore

$$a^3+b^3=(a+b)^3-3ab(a+b)=7^3-3\cdot 5\cdot 7=\boxed{238}.$$

Solving $t^2-7t+5=0$ first would introduce irrational roots that cancel at the end. Avoiding those roots is the advantage of symmetry. □

Example 3.2 (A complicated expression in disguise)**.** Given $x+y=3$ and $xy=1$, find

$$x^4-5x^3y-5xy^3+y^4.$$

Solution. The four terms are arranged symmetrically, so it is reasonable to search for a formula in $s=x+y$ and $p=xy$. Grouping the terms gives

$$\begin{aligned} x^4-5x^3y-5xy^3+y^4 &= (x^4+y^4)-5xy(x^2+y^2)\\ &= (s^4-4ps^2+2p^2)-5p(s^2-2p)\\ &= s^4-9ps^2+12p^2. \end{aligned}$$

Substituting $s=3$ and $p=1$, the expression becomes $81-81+12=\boxed{12}$. This more elaborate identity is featured in the Randolph Math League *Part II* notes; the derivation matters much more than the final formula. □

3.1 When do the actual variables matter?

Knowing s and p is equivalent to knowing that x and y are the two roots of

$$t^2 - st + p = 0.$$

This is a powerful connection, but it also provides a warning. If x, y must be real, the discriminant must be nonnegative:

$$s^2 - 4p = (x - y)^2 \geq 0.$$

Further, knowing only $(x + y)^2$ does not necessarily determine $x + y$: both signs may occur. Many otherwise good contest solutions lose an answer by silently taking a positive square root.

4 Power Sums and Recurrences

If a problem asks for $x^{12} + y^{12}$, expanding $(x + y)^{12}$ is rarely a good idea. The right approach is to exploit the fact that the same two numbers x and y appear in every term. Let

$$P_n = x^n + y^n, \qquad s = x + y, \quad p = xy.$$

Multiplying P_{n-1} by s gives

$$sP_{n-1} = (x + y)(x^{n-1} + y^{n-1}) = x^n + y^n + xy(x^{n-2} + y^{n-2}).$$

The additional terms are precisely pP_{n-2}. Therefore, for $n \geq 2$,

$$\boxed{P_n = sP_{n-1} - pP_{n-2}, \qquad P_0 = 2,\ P_1 = s.}$$

The recurrence tells us that every power sum can be calculated from the preceding two. This is an elementary instance of the same algebraic structure that appears in Newton's identities and linear recurrence sequences. On a contest, it converts an expression with huge exponents into repeated arithmetic on small numbers.

Example 4.1 (Sixth powers without radicals)**.** Suppose $x + y = 3$ and $xy = 1$. Find $x^6 + y^6$.

Solution. Here $P_n = 3P_{n-1} - P_{n-2}$. Starting at $P_0 = 2$ and $P_1 = 3$,

n	0	1	2	3	4	5	6
P_n	2	3	7	18	47	123	322

Hence the requested value is $\boxed{322}$. At no stage did we need the individual roots of $t^2 - 3t + 1 = 0$. □

Example 4.2 (A recurrence hidden inside a quadratic)**.** Let u, v be the two roots of $t^2 - 4t + 1 = 0$. Compute $u^5 + v^5$.

Solution. The sum of the roots is 4 and their product is 1. Thus $P_n = 4P_{n-1} - P_{n-2}$. Starting from $P_0 = 2$ and $P_1 = 4$, we obtain $P_2 = 14$, $P_3 = 52$, $P_4 = 194$, and $P_5 = \boxed{724}$. The quadratic supplies the recurrence directly, so extracting its roots would be a detour. □

5 Reciprocal Expressions

Expressions containing both x and $1/x$ often look more complicated than ordinary polynomials. In fact, they are usually *more* structured, because the two quantities have product 1. For $x \neq 0$, introduce

$$t = x + \frac{1}{x}.$$

Then

$$x^2+\frac{1}{x^2}=t^2-2, \qquad x^3+\frac{1}{x^3}=t^3-3t, \qquad x^4+\frac{1}{x^4}=t^4-4t^2+2.$$

These expressions obey the recurrence from the previous section with $s=t$ and $p=1$. The substitution also makes the symmetry $x \leftrightarrow 1/x$ explicit.

Example 5.1 (Why the sign matters)**.** A nonzero real number x satisfies $x^2+x^{-2}=7$. Determine *all* possible values of x^3+x^{-3}. *(Motivated by a problem in Randolph Math League, 2013–14.)*

Solution. Set $t=x+x^{-1}$. The given becomes $t^2-2=7$, so $t^2=9$ and $t=\pm 3$. Therefore

$$x^3+x^{-3}=t^3-3t=\begin{cases}18, & t=3,\\ -18, & t=-3.\end{cases}$$

Both possibilities occur, since the equations $x+1/x=3$ and $x+1/x=-3$ each have nonzero real solutions. Thus the answer is $\boxed{18 \text{ or } -18}$. Assuming $t=3$ without knowing $x>0$ would discard a valid case. □

5.1 Multiplying mixed reciprocal expressions

Some problems use two different variables, but the right multiplication still creates a reciprocal pair. This is a case where the target expression itself suggests the operation.

Example 5.2 (The multiplication that saves a page)**.** Positive real numbers a, b satisfy

$$a+\frac{1}{b}=7, \qquad b+\frac{1}{a}=5.$$

Find $ab+1/(ab)$. *(Adapted from the motivating example in Randolph Math League, Part II.)*

Solution. Multiplying the givens gives

$$35=\left(a+\frac{1}{b}\right)\left(b+\frac{1}{a}\right)=ab+1+1+\frac{1}{ab}.$$

The middle terms are both 1, so the target appears with no need to determine either variable:

$$ab+\frac{1}{ab}=\boxed{33}.$$

A conventional solution for a and b produces cumbersome radicals. Multiplying the equations is not magic; the requested expression has already told us which cross terms will be useful. □

6 Three-Variable Symmetry and Cubic Identities

For three variables, the most natural symmetric quantities are

$$s=a+b+c, \qquad q=ab+bc+ca, \qquad r=abc.$$

One can think of these as the information about a, b, c that survives when their order is forgotten. Many apparently lengthy calculations become short after they are rewritten in terms of s, q, r:

$$\begin{aligned} a^2+b^2+c^2 &= s^2-2q,\\ a^3+b^3+c^3 &= s^3-3sq+3r,\\ (a+b)(b+c)(c+a) &= sq-r. \end{aligned}$$

The first identity comes from squaring s. To understand the third, expand $(a+b)(b+c)(c+a)$ and group its terms by how often each variable appears; the six mixed cubic terms are exactly those in sq, except that abc has been counted one time too many.

A particularly useful result is

$$\begin{aligned} a^3+b^3+c^3-3abc &= (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \\ &= \frac{a+b+c}{2}\big((a-b)^2+(b-c)^2+(c-a)^2\big). \end{aligned}$$

This identity connects two themes at once. It gives the remarkable simplification $a^3+b^3+c^3=3abc$ whenever $a+b+c=0$. It also shows that for real a,b,c with nonnegative sum, $a^3+b^3+c^3\geq 3abc$; a factorization doubles as an inequality.

Example 6.1 (A cubic sum from three small invariants)**.** Suppose $a+b+c=6$, $ab+bc+ca=11$, and $abc=6$. Find $a^3+b^3+c^3$ and $(a+b)(b+c)(c+a)$.

Solution. We have $s=6$, $q=11$, and $r=6$. Substitution into the cubic sum identity gives

$$a^3+b^3+c^3=6^3-3(6)(11)+3(6)=\boxed{36}.$$

Similarly,

$$(a+b)(b+c)(c+a)=6(11)-6=\boxed{60}.$$

We could infer that a,b,c are $1,2,3$ in some order, but neither requested expression requires us to find that ordering. □

Example 6.2 (The zero-sum shortcut)**.** Real numbers u,v,w satisfy $u+v+w=0$ and $uvw=-12$. Determine $u^3+v^3+w^3$.

Solution. In the identity

$$u^3+v^3+w^3-3uvw=(u+v+w)(u^2+v^2+w^2-uv-vw-wu),$$

the entire right side vanishes. Hence $u^3+v^3+w^3=3uvw=\boxed{-36}$. The important clue is the zero sum, not the numerical value of the product. □

7 Systems of Equations: Solve Only What Is Asked

A large system is not necessarily a hard system. When several equations have a common pattern, a carefully chosen sum or difference can reveal the requested expression without conventional elimination. This is particularly effective when all but one coefficient are identical in every equation.

Example 7.1 (A common total hiding in every equation)**.** Let $S=x_1+x_2+x_3+x_4$, and suppose

$$S+x_1=6,\quad S+x_2=12,\quad S+x_3=18,\quad S+x_4=24.$$

Find x_3+2x_4.

Solution. Adding all four equations produces $4S+(x_1+x_2+x_3+x_4)=60$, or $5S=60$. Thus $S=12$. We can now read off the variables we actually need: $x_3=18-12=6$ and $x_4=24-12=12$. Therefore $x_3+2x_4=\boxed{30}$. There was no reason to solve the first two variables separately. □

Example 7.2 (A five-variable AIME pattern)**.** Suppose five real numbers satisfy

$$(x_1+\cdots+x_5)+x_i=6\cdot 2^{i-1}\quad \text{for } i=1,2,3,4,5.$$

Compute $3x_4 + 2x_5$. *(This is the system structure used in the 1986 AIME problem reproduced in Randolph Math League, 2013–14.)*

Solution. Write $S = x_1 + \cdots + x_5$. Summing the five equations gives

$$6S = 6(1 + 2 + 4 + 8 + 16) = 186, \qquad S = 31.$$

Consequently $x_4 = 48 - 31 = 17$ and $x_5 = 96 - 31 = 65$, leading to

$$3x_4 + 2x_5 = 3(17) + 2(65) = \boxed{181}.$$

This calculation captures the general lesson of the example: if every equation is a common expression plus a small adjustment, determine the common expression first. □

7.1 Naming a repeated expression

In a nonlinear-looking system, the repeated object might not be a linear sum on its own. For example, if each equation includes $x + y + z$, writing $S = x + y + z$ immediately makes it possible to add the equations. The natural variable is often an expression, not one of the original letters.

Example 7.3 (A system with two sign-related solutions)**.** Real numbers x, y, z satisfy

$$(x + y)(x + y + z) = 40, \qquad (y + z)(x + y + z) = 70, \qquad (z + x)(x + y + z) = 90.$$

Find every ordered triple (x, y, z).

Solution. Set $S = x + y + z$. The sum of the equations is

$$\big((x + y) + (y + z) + (z + x)\big)S = 200,$$

so $2S^2 = 200$ and $S = \pm 10$. If $S = 10$, then $x+y = 4$, $y+z = 7$, and $z+x = 9$, giving $(x, y, z) = (3, 1, 6)$. If $S = -10$, each pairwise sum changes sign, giving $(-3, -1, -6)$. Thus

$$\boxed{(x, y, z) = (3, 1, 6) \text{ or } (-3, -1, -6)}.$$

The two solutions make sense: simultaneously negating all three variables leaves the original products unchanged. □

8 Inverting Fractions and Changing Variables

Fractions can be deceptively nonlinear. Suppose a problem gives $ab/(a + b)$; the numerator and denominator both involve two variables, and direct substitution may seem hopeless. Taking the reciprocal reveals a simple sum:

$$\frac{a + b}{ab} = \frac{1}{a} + \frac{1}{b}.$$

This suggests setting $u = 1/a$, $v = 1/b$, and perhaps $w = 1/c$. A system involving products divided by sums then becomes a system of pairwise linear sums. This substitution is legitimate whenever the relevant denominators are nonzero, a condition that should always be checked.

Example 8.1 (A nonlinear system becomes three linear equations)**.** Nonzero real numbers a, b, c satisfy

$$\frac{ab}{a + b} = 3, \qquad \frac{bc}{b + c} = 4, \qquad \frac{ca}{c + a} = 5.$$

Find $\dfrac{abc}{ab+bc+ca}$. *(Selected from Randolph Math League, 2013–14.)*

Solution. The target itself suggests reciprocals:

$$\frac{abc}{ab+bc+ca} = \frac{1}{1/a+1/b+1/c}.$$

Taking the reciprocal of each given yields

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{3}, \quad \frac{1}{b}+\frac{1}{c}=\frac{1}{4}, \quad \frac{1}{c}+\frac{1}{a}=\frac{1}{5}.$$

Adding and dividing by two,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)=\frac{47}{120}.$$

The requested quantity is therefore $\boxed{\dfrac{120}{47}}$. Notice that we never had to find a, b, c. □

Another useful substitution appears when an expression contains both $x+y$ and $x-y$. Setting $s = x+y$ and $d = x-y$ turns x^2-y^2 into sd. The best substitution is the one that turns the target into something simple, not necessarily the one that simplifies every line of the givens.

9 More Subtle Patterns: Conjugates and Cyclic Products

Some of the strongest AIME algebra problems offer no obvious sum or product at first glance. Two additional patterns are especially useful: a radical expression whose conjugate is its reciprocal, and a cyclic product that becomes symmetric after clearing denominators.

9.1 A radical can satisfy a much simpler equation

Let $t = \sqrt{3}+\sqrt{2}$. Its conjugate is $\sqrt{3}-\sqrt{2}$, and

$$(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) = 1.$$

Hence the conjugate is $1/t$. Also,

$$t^2 = 5+2\sqrt{6}, \qquad (t^2-5)^2 = 24, \qquad t^4-10t^2+1=0.$$

Because $t \neq 0$, dividing the last relation by t gives

$$\frac{1}{t} = 10t - t^3.$$

Thus a nonlinear-looking conjugation can be performed using the polynomial $10x - x^3$. This example comes from the Randolph Math League *Part II* notes. The general technique is to establish an algebraic relation for a radical, then use that relation to replace a reciprocal or a high power.

9.2 A cyclic product that secretly factors

Consider the identity

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right) = \left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right),$$

where a, b, c are nonzero. Multiplying by abc yields

$$(1+ab)(1+bc)(1+ca) = (1+a)(1+b)(1+c).$$

Write $s = a+b+c$, $q = ab+bc+ca$, and $r = abc$. The left side expands to $1+q+rs+r^2$, while the right side becomes $1+s+q+r$. Subtracting shows that the complicated equality is exactly

$$\boxed{(r-1)(r+s) = 0.}$$

This factorization deserves special attention because it has *two* branches: either $abc = 1$, in which case the original equality places no further restriction on $a+b+c$, or $a+b+c = -abc$. One must not divide by $abc-1$ unless it is known to be nonzero. For example, if $abc = 13$, the sum must be -13. The corresponding observation is discussed in the Randolph Math League *Part II* lecture.

Important Observation. The most interesting line in a strong algebra solution is often not a calculation but an interpretation: *"The expression is symmetric," "both sides have the same total," "the conjugate is a reciprocal,"* or *"this apparent cubic equation factors into two cases."* Learning to articulate that observation helps make the next step natural rather than mysterious.

10 Problem Set

These problems are deliberately arranged in stages. The introductory set reinforces identities that should become immediate; the intermediate set rewards a well-chosen substitution; the advanced set mixes several ideas; and the final challenges resemble the sort of algebraic recognition that can make an AIME problem dramatically shorter. Problems explicitly labeled *Randolph*, *AIME*, or *NIMO* are selected or adapted from the cited source handouts; the other formulations were written for this guide.

10.1 Introductory AMC 10 Problems

1. If $x+y = 11$ and $xy = 24$, find x^2+y^2.

2. Real numbers a, b satisfy $a+b = 8$ and $ab = 12$. Find a^3+b^3.

3. If $x+y = 7$ and $xy = 10$, compute $(x+2)(y+2)$.

4. A nonzero real number x satisfies $x + 1/x = 4$. Find $x^2 + 1/x^2$.

5. Suppose $a+b+c = 9$ and $a^2+b^2+c^2 = 35$. Find $ab+bc+ca$.

6. If $x-y = \sqrt{5}$ and $x+y = \sqrt{20}$, evaluate $(x^2-y^2)^2$.

7. Real numbers a, b satisfy $a+b = 8$ and $ab = 12$. Find $(a-1)(b-1)$.

8. Nonzero real numbers p, q satisfy $p+q = 6$ and $1/p + 1/q = 3/2$. Find p^2+q^2. *(Randolph 2013–14, adapted.)*

9. Real numbers x, y satisfy $x^2+y^2 = 29$ and $xy = 10$. Determine all possible values of $x+y$.

10.2 Intermediate AMC 10/12 Problems

10. If $x+y = 5$ and $xy = 4$, find x^5+y^5 without finding x or y.

11. A nonzero real number x satisfies $x^2 + x^{-2} = 7$. Determine all values of $x^3 + x^{-3}$. *(Randolph 2013–14, adapted.)*

12. Positive real numbers a, b satisfy $a + 1/b = 6$ and $b + 1/a = 4$. Find $ab + 1/(ab)$.

13. If $a + b = 3$ and $a^2 + b^2 = 5$, find $a^4 + b^4$.

14. Let $x + y + z = 7$, $xy + yz + zx = 14$, and $xyz = 8$. Compute $(x + y)(y + z)(z + x)$.

15. Real numbers u, v, w satisfy $u + v + w = 0$ and $uvw = -6$. Find $u^3 + v^3 + w^3$.

16. Nonzero real numbers a, b, c satisfy $ab/(a + b) = 3$, $bc/(b + c) = 4$, and $ca/(c + a) = 5$. Find $abc/(ab + bc + ca)$. *(Randolph 2013–14, selected.)*

17. Let $S = x_1 + x_2 + x_3 + x_4$. If $S + x_i = 6i$ for $i = 1, 2, 3, 4$, compute $x_3 + 2x_4$.

18. Suppose $xy = 6$ and $x^2y + xy^2 + x + y = 63$. Determine $x^2 + y^2$. *(AHSME 1987, as listed by Randolph.)*

19. Real x, y, z satisfy $(x + y)S = 40$, $(y + z)S = 70$, and $(z + x)S = 90$, where $S = x + y + z$. Find every ordered triple (x, y, z).

10.3 Advanced AMC 12 / AIME Preparation

20. ★ If $a - b = 2$ and $a^3 - b^3 = 26$, find all possible values of $a + b$.

21. ★ Let x be a nonzero real number satisfying $x + 1/x = 3$. Compute $x^8 + x^{-8}$.

22. ★ If $x + y = 5$ and $xy = 5$, evaluate $x^7 + y^7$ using a recurrence.

23. ★ Suppose $a + b + c = 6$, $ab + bc + ca = 11$, and $abc = 6$. Find $a^3 + b^3 + c^3$.

24. ★ Real numbers x, y satisfy $x + y = 5$ and $x^3 + y^3 = 35$. Find $x^6 + y^6$.

25. ★ If $a + b = 5$ and $ab = 6$, compute $\dfrac{a^4 - b^4}{a - b}$.

26. ★ Real numbers x, y satisfy $x^2 + y^2 = 10$ and $x^4 + y^4 = 82$. Find every possible value of xy.

27. ★ Positive real numbers a, b, c satisfy $a^2 + b^2 + c^2 = 989$ and $(a + b)^2 + (b + c)^2 + (c + a)^2 = 2013$. Determine $a + b + c$. *(Randolph 2013–14, selected.)*

28. ★ Suppose $a + b + c = 10$, $a^2 + b^2 + c^2 = 38$, and $abc = 30$. Find $(a + b)(b + c)(c + a)$.

10.4 Challenge Problems and Contest Connections

29. ★ Let $t = \sqrt{3} + \sqrt{2}$. Find a polynomial $Q(x)$ with integer coefficients and degree at most three such that $Q(t) = \sqrt{3} - \sqrt{2}$. *(Randolph Part II, adapted.)*

30. ★ Nonzero real numbers a, b, c satisfy

$$\left(a + \frac{1}{b}\right)\left(b + \frac{1}{c}\right)\left(c + \frac{1}{a}\right) = \left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right).$$

If $abc = 13$, find $a + b + c$. Also explain what changes when $abc = 1$. *(Randolph Part II, adapted.)*

31. ★ Positive real numbers a, b satisfy $ab = 2$ and

$$\frac{a}{a+b^2} + \frac{b}{b+a^2} = \frac{7}{8}.$$

Find $a^6 + b^6$. *(NIMO 2014, listed in Randolph Part II.)*

32. ★ Five real numbers $x_1, \ldots, x_5$ satisfy $S + x_i = 6 \cdot 2^{i-1}$ for $i = 1, \ldots, 5$, where $S = x_1 + \cdots + x_5$. Find $3x_4 + 2x_5$. *(AIME 1986, listed in Randolph 2013–14.)*

33. ★ Real numbers a, b, c satisfy $a + b + c = 9$ and $a^2 + b^2 + c^2 = 29$. Give *two examples* of positive triples satisfying the conditions but having different products abc. Explain what this reveals about symmetric information.

34. ★ Nonzero real numbers a, b, c satisfy

$$\frac{1}{a} + \frac{1}{b} = \frac{5}{6}, \qquad \frac{1}{b} + \frac{1}{c} = \frac{1}{2}, \qquad \frac{1}{c} + \frac{1}{a} = \frac{2}{3}.$$

Compute $\dfrac{abc}{ab + bc + ca}$ and $\dfrac{a + b + c}{abc}$ without finding a, b, c.

11 Solutions and Explanations

11.1 Introductory Solutions

1. The square $(x + y)^2$ contains both $x^2 + y^2$ and $2xy$. Removing the latter gives

$$x^2 + y^2 = (x + y)^2 - 2xy = 121 - 48 = \boxed{73}.$$

2. The cube identity is more direct than solving for a, b:

$$a^3 + b^3 = (a + b)^3 - 3ab(a + b) = 512 - 288 = \boxed{224}.$$

3. Expand the requested product, not the two variables:

$$(x + 2)(y + 2) = xy + 2(x + y) + 4 = 10 + 14 + 4 = \boxed{28}.$$

4. Square the given identity. Since $(x + 1/x)^2 = x^2 + 2 + x^{-2}$, the answer is $16 - 2 = \boxed{14}$.

5. The three pairwise products appear when the total sum is squared:

$$81 = 35 + 2(ab + bc + ca), \qquad ab + bc + ca = \boxed{23}.$$

6. Factor the difference of squares before squaring it:

$$(x^2 - y^2)^2 = ((x - y)(x + y))^2 = 5 \cdot 20 = \boxed{100}.$$

7. Since $(a - 1)(b - 1) = ab - (a + b) + 1$, its value is $12 - 8 + 1 = \boxed{5}$.

8. The reciprocals combine as $(p + q)/(pq) = 3/2$. Thus $6/(pq) = 3/2$, so $pq = 4$ and $p^2 + q^2 = 6^2 - 2(4) = \boxed{28}$.

9. We know $(x + y)^2 = x^2 + y^2 + 2xy = 29 + 20 = 49$. Therefore $x + y = \boxed{\pm 7}$. Both signs are possible: $(x, y) = (5, 2)$ and $(-5, -2)$ meet the given conditions. The sign is not determined by symmetric even-degree information alone.

11.2 Intermediate Solutions

10. Let $P_n = x^n + y^n$. With $s = 5$ and $p = 4$, the recurrence is $P_n = 5P_{n-1} - 4P_{n-2}$. Starting at $P_0 = 2$, $P_1 = 5$, we find $P_2 = 17$, $P_3 = 65$, $P_4 = 257$, and $P_5 = \boxed{1025}$. These are the powers of the pair $\{1, 4\}$, but the recurrence finds the result without identifying that pair.

11. Put $t = x + x^{-1}$. From $x^2 + x^{-2} = t^2 - 2 = 7$, we obtain $t = \pm 3$. The cubic reciprocal identity gives $x^3 + x^{-3} = t^3 - 3t$, so the answers are $\boxed{18 \text{ and } -18}$. Both signs are attainable by nonzero real x.

12. Multiplying the two given equations immediately creates the target:

$$24 = \left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = ab + 2 + \frac{1}{ab}.$$

The answer is $\boxed{22}$. There is no reason to solve a quadratic for either variable.

13. First recover the product: $9 = 5 + 2ab$, so $ab = 2$. Now treat a^2, b^2 as a pair:

$$a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2 = 25 - 8 = \boxed{17}.$$

14. Use $(x + y)(y + z)(z + x) = (x + y + z)(xy + yz + zx) - xyz$. Substitution gives $7 \cdot 14 - 8 = \boxed{90}$.

15. Because $u+v+w = 0$, the identity $u^3+v^3+w^3-3uvw = (u+v+w)(u^2+v^2+w^2-uv-vw-wu)$ reduces to $u^3 + v^3 + w^3 = 3uvw = \boxed{-18}$.

16. Invert each condition to obtain $1/a + 1/b = 1/3$, $1/b + 1/c = 1/4$, and $1/c + 1/a = 1/5$. Their sum is $2(1/a + 1/b + 1/c) = 47/60$, so $1/a + 1/b + 1/c = 47/120$. Since

$$\frac{abc}{ab + bc + ca} = \frac{1}{1/a + 1/b + 1/c},$$

the answer is $\boxed{120/47}$.

17. Summing $S + x_i = 6i$ over $i = 1, \ldots, 4$ gives $5S = 6(1 + 2 + 3 + 4) = 60$, so $S = 12$. Thus $x_3 = 18 - 12 = 6$ and $x_4 = 24 - 12 = 12$. The target is $6 + 2(12) = \boxed{30}$.

18. Factor the second expression as

$$x^2y + xy^2 + x + y = xy(x + y) + (x + y) = (xy + 1)(x + y).$$

Because $xy = 6$, we have $7(x + y) = 63$, or $x + y = 9$. Consequently $x^2 + y^2 = 9^2 - 2(6) = \boxed{69}$. The factoring step is exactly what the numerical values were designed to reward.

19. Add all three equations. As $(x + y) + (y + z) + (z + x) = 2S$, we obtain $2S^2 = 200$ and $S = \pm 10$. For $S = 10$, the pairwise sums are $4, 7, 9$, giving $(x, y, z) = (3, 1, 6)$. For $S = -10$, they are $-4, -7, -9$, giving $(-3, -1, -6)$. The complete answer is

$$\boxed{(3, 1, 6) \text{ or } (-3, -1, -6)}.$$

11.3 Advanced Solutions

20. Factoring $a^3 - b^3$ and dividing by $a - b = 2$ gives

$$a^2 + ab + b^2 = 13.$$

Let $s = a + b$ and $d = a - b = 2$. From $a = (s + d)/2$, $b = (s - d)/2$, we derive

$$a^2 + ab + b^2 = \frac{3s^2 + d^2}{4}.$$

Hence $13 = (3s^2 + 4)/4$, so $s^2 = 16$ and $a + b = \boxed{\pm 4}$. Both occur: $(a, b) = (3, 1)$ and $(-1, -3)$ satisfy the givens. The two signs are not an algebraic nuisance; they are genuine solutions.

21. Set $R_n = x^n + x^{-n}$. Because $x + 1/x = 3$, the recurrence is $R_n = 3R_{n-1} - R_{n-2}$ with $R_0 = 2$ and $R_1 = 3$. The successive values are $R_2 = 7$, $R_3 = 18$, $R_4 = 47$, $R_5 = 123$, $R_6 = 322$, $R_7 = 843$, and $R_8 = \boxed{2207}$. Notice that a high power has required only integer arithmetic.

22. With $s = 5$ and $p = 5$, let $P_n = x^n + y^n$. Starting with $P_0 = 2$, $P_1 = 5$ and repeatedly using $P_n = 5P_{n-1} - 5P_{n-2}$ gives

$$P_2 = 15,\ P_3 = 50,\ P_4 = 175,\ P_5 = 625,\ P_6 = 2250,\ P_7 = \boxed{8125}.$$

The recurrence is much cleaner than substituting the radical roots of $t^2 - 5t + 5 = 0$.

23. The three-variable cube identity states

$$a^3 + b^3 + c^3 = s^3 - 3sq + 3r,$$

where $s = a+b+c$, $q = ab+bc+ca$, and $r = abc$. Plugging in the given values gives $216-198+18 = \boxed{36}$.

24. The cube identity allows us to recover a missing invariant: $35 = 125 - 3(xy)(5)$, hence $xy = 6$. Rather than calculate x and y, regard x^3, y^3 as the two numbers in a new sum-of-squares problem:

$$x^6 + y^6 = (x^3 + y^3)^2 - 2(xy)^3 = 35^2 - 2(6^3) = \boxed{793}.$$

This two-stage strategy is common on harder AMC questions: use one identity to extract the ingredient required by another.

25. The given sum and product also determine $a^2 + b^2 = (a + b)^2 - 2ab = 25 - 12 = 13$. Since $a \neq b$ (indeed the two values are 2 and 3), cancel $a - b$ in the difference of fourth powers:

$$\frac{a^4 - b^4}{a - b} = (a + b)(a^2 + b^2) = 5 \cdot 13 = \boxed{65}.$$

The key was to factor before substituting.

26. Treat x^2 and y^2 as a pair. Their sum is 10, so

$$82 = (x^2 + y^2)^2 - 2x^2y^2 = 100 - 2(xy)^2.$$

Thus $(xy)^2 = 9$ and $xy = \boxed{\pm 3}$. Both occur: $(x, y) = (3, 1)$ gives $xy = 3$, whereas $(3, -1)$ gives $xy = -3$, and both pairs satisfy the even-power conditions.

27. Expanding the three pairwise squares gives

$$(a + b)^2 + (b + c)^2 + (c + a)^2 = 2(a^2 + b^2 + c^2) + 2(ab + bc + ca).$$

Therefore $2013 = 1978 + 2q$, so $q = 35/2$. The total square becomes

$$(a + b + c)^2 = 989 + 2q = 989 + 35 = 1024.$$

Because a, b, c are positive, their sum is positive; hence $a + b + c = \boxed{32}$. The positivity assumption is what selects the sign of the square root.

28. From $s = 10$ and $a^2 + b^2 + c^2 = 38$, we obtain $q = (s^2 - 38)/2 = 31$. The pairwise-sum product identity then gives

$$(a + b)(b + c)(c + a) = sq - r = 10(31) - 30 = \boxed{280}.$$

This calculation is especially efficient because the target is symmetric in all three variables.

11.4 Challenge Solutions

29. The product $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})=1$ shows that the desired conjugate equals $1/t$. Now

$$t^2 = 5+2\sqrt{6} \quad\Longrightarrow\quad (t^2-5)^2 = 24 \quad\Longrightarrow\quad t^4-10t^2+1=0.$$

Dividing by $t\neq 0$ yields $t^3-10t+t^{-1}=0$. Thus

$$\frac{1}{t} = 10t - t^3, \qquad \boxed{Q(x)=10x-x^3}.$$

The problem asks for a polynomial of degree at most three, not necessarily a proof that no other polynomial could agree at this single algebraic number. The displayed polynomial meets the requirement exactly.

30. Multiply the equation by abc, which is nonzero. It becomes

$$(1+ab)(1+bc)(1+ca) = (1+a)(1+b)(1+c).$$

Writing $s=a+b+c$, $q=ab+bc+ca$, and $r=abc$, the two sides are respectively $1+q+rs+r^2$ and $1+s+q+r$. Hence

$$(r-1)(r+s)=0.$$

When $r=13$, the first factor is nonzero, so $s=-r=\boxed{-13}$. If $abc=1$, however, the identity is automatic and does *not* determine $a+b+c$. The special case is why we must factor rather than divide by $r-1$ without checking it.

31. Combine the two fractions before considering the individual variables:

$$\begin{aligned}\frac{a}{a+b^2}+\frac{b}{b+a^2} &= \frac{a(b+a^2)+b(a+b^2)}{(a+b^2)(b+a^2)}\\ &= \frac{a^3+b^3+2ab}{a^3+b^3+ab+a^2b^2}.\end{aligned}$$

Let $T=a^3+b^3$. Since $ab=2$, the equation reduces to $(T+4)/(T+6)=7/8$. Cross-multiplying gives $8T+32=7T+42$, so $T=10$. Now

$$a^6+b^6=(a^3+b^3)^2-2a^3b^3=T^2-2(ab)^3=100-16=\boxed{84}.$$

The insight is that both denominators expand into the *same* symmetric cubic quantity; solving for a,b would have hidden this structure.

32. Put $S=x_1+x_2+x_3+x_4+x_5$. Adding the five equations gives

$$6S = 6(1+2+4+8+16) = 186, \qquad S=31.$$

Thus $x_4=6\cdot 8-S=17$ and $x_5=6\cdot 16-S=65$. The requested linear combination is $3(17)+2(65)=\boxed{181}$. The five-variable notation conceals a single unknown total; once that total is known, only two of the variables need to be evaluated.

33. One positive triple is $(2,3,4)$: its sum is 9, its sum of squares is $4+9+16=29$, and its product is 24. A different positive triple is

$$\left(\frac{5}{2}, \frac{13+\sqrt{13}}{4}, \frac{13-\sqrt{13}}{4}\right).$$

The latter two entries sum to $13/2$ and have product $(169-13)/16 = 39/4$. Consequently the full triple has sum 9 and sum of squares

$$\frac{25}{4} + \left(\frac{13}{2}\right)^2 - 2\left(\frac{39}{4}\right) = 29,$$

but its product is $(5/2)(39/4) = 195/8 \neq 24$. Thus the first two symmetric quantities $s = a+b+c$ and $q = ab+bc+ca$ do not generally determine the third one, $r = abc$. The equations constrain the triple without fixing it uniquely.

34. The requested first expression is the reciprocal of the total reciprocal sum:

$$\frac{abc}{ab+bc+ca} = \frac{1}{1/a+1/b+1/c}.$$

Adding the three given equations gives

$$2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = \frac{5}{6}+\frac{1}{2}+\frac{2}{3} = 2,$$

so $1/a+1/b+1/c = 1$, and the first answer is $\boxed{1}$. The second target is

$$\frac{a+b+c}{abc} = \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}.$$

Here the three pairwise reciprocal sums actually determine $1/a, 1/b, 1/c$. Let $u = 1/a$, $v = 1/b$, $w = 1/c$. Subtracting the second given from the sum of the first and third yields $2u = 5/6 + 2/3 - 1/2 = 1$, so $u = 1/2$. Then $v = 5/6 - 1/2 = 1/3$ and $w = 1/2 - 1/3 = 1/6$. Therefore

$$\frac{a+b+c}{abc} = uv+vw+wu = \frac{1}{6}+\frac{1}{18}+\frac{1}{12} = \boxed{\frac{11}{36}}.$$

In this final problem, the first target requires only the total reciprocal sum, while the second requires the pairwise reciprocal products. Matching the amount of work to the target is the whole point of the guide.

12 Closing Perspective

When a manipulation seems resistant to the original approach, don't immediately resort to a longer calculation. Look again at the requested expression and keep your eyes on the prize. Is it unchanged when two variables are interchanged? Does it resemble a difference of squares, a sum of cubes, or a shifted product? Could a single named sum replace repeated groups of variables? Would reciprocals transform the system into pairwise sums? Could a power recurrence replace radical arithmetic?

Sources and Further Reading

1. Randolph High School Math League, *Algebraic Manipulations* (2013–2014), distributed through Art of Problem Solving. Selected problems and contest attributions are identified in the text.

2. Randolph High School Math League, *Algebraic Manipulations Part II* (2014–2015), hosted by David Altizio: `https://davidaltizio.web.illinois.edu/Algebraic%20Manipulations%20Part%20II.pdf`.

3. Aryan Ayyanger, *Polynomials on the AMC 10/12: Guide #1 — Roots, Vieta, and Transformations*: `https://www.betweenthesteps.com/Polynomials%20Guide%201.pdf`.

4. Aryan Ayyanger, *Triangles on the AMC 10/12: Guide 1 — Area, Lengths, Angles.* Layout and progression used as a formatting reference for this guide.